An identity involving counting sums of square and triangular numbers

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One of the exercises in [1, (3.6.2), p.71] is to prove

$$\phi(q)\psi(q^2) = \left[\psi(q)\right]^2, \tag{1}$$

where ϕ and ψ are generating functions encoding square and triangular numbers respectively:

$$\begin{aligned} \phi(q) &= \sum_{r=-\infty}^{\infty} q^{r^2} &&= \cdots + q^9 + q^4 + q + 1 + q + q^4 + q^9 + \cdots ; \\ \psi(q) &= \sum_{i=0}^{\infty} q^{i(i+1)/2} &&= 1 + q + q^3 + q^6 + q^{10} + \cdots . \end{aligned}$$

The reason for the choice of r and i as index variables will become clear below.

The expression $i(i+1)/2$, for $i \geq 0$, is the ith triangular number:

$$T_i = \frac{i(i+1)}{2}.$$

The result (1) has an interpretation in terms of sums of square and triangular numbers, which we see by considering the coefficient of q^n on each side:

Theorem The number of ways of expressing an integer n as the sum of a square and twice a triangular number is the same as the number of ways of expressing n as the sum of two triangular numbers. More formally, the following two counts are equal:

A square plus twice a triangular number The number of ordered pairs (r, i) such that

$$r \in \mathbb{Z}; \quad i \in \mathbb{Z}_{\geq 0}; \quad n = r^2 + 2T_i.$$

In particular note that, when $r \neq 0$, the two pairs (r, i) and $(-r, i)$ are distinct, even though they give rise to the same sum $r^2 + 2T_i$.

Two triangular numbers The number of ordered pairs (j, k) such that

$$j \in \mathbb{Z}_{\geq 0}; \quad k \in \mathbb{Z}_{\geq 0}; \quad n = T_j + T_k.$$

In a similar vein, note that if $j \neq k$, the two pairs (j, k) and (k, j) are distinct, even though they give rise to the same two summands, T_j and T_k.

Bijective proof

A bijective proof of the identity (1) consists of showing that for every sum $n = r^2 + T_i$, we can find a corresponding $n = T_j + T_k$ and vice versa. We now give such a proof.

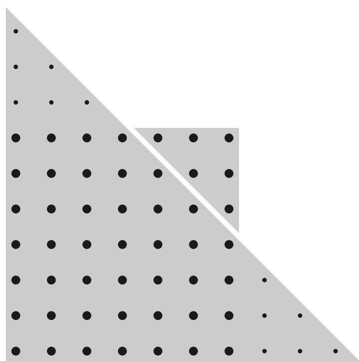
We first deal with the case that $r = 0$. Here, we take $j = k = i$ and have $T_j + T_k = 2T_i = n$ as required.

For $r \neq 0$, we will work with $|r|$ and find a pair (j, k) with $j > k$. If $r > 0$, then our final pair will be this (j, k), but if $r < 0$, we will instead reverse the pair and take (k, j).

Outline of proof

Pictorially, the idea is to draw two copies of the triangle of dots representing T_i, one abutting each of two adjacent faces of a square of dots representing r^2. The total number of dots drawn will therefore be n, and we then decompose the dots into two triangles. We find that we must analyse two cases separately:

'Large' square



First, if $|r| > i$, we have a situation as shown to the right (where $|r| = 7$ and $i = 3$). We see that we can re-form the dots into two triangles, with sides of $|r| + i$ dots and $|r| - (i + 1)$ dots. I.e., we take:

$$j = |r| + i; \quad k = |r| - (i + 1).$$

It is straightforward to verify that

$$T_j + T_k = r^2 + 2T_i.$$

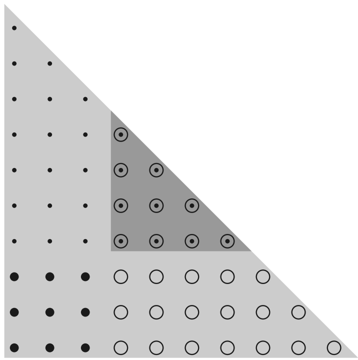
Because we are currently considering only $|r| > i$, i.e., $|r| \geq i + 1$, we see $k \geq 0$ as required. We also have

$$j - k = 2i + 1,$$

which is strictly positive because $i \geq 0$, and so $j > k$. Note also that $j - k$ is odd.

‘Small’ square

On the other hand (while still within the case $r \neq 0$), if $|r| \leq i$, we (usually) find that the triangles ‘overlap’ as shown (in the figure, $r = 3$ and $i = 7$). Only by counting the double-struck dots (drawn as $\odot$) twice do we have n dots altogether. We can instead view the figure as two nested triangles as shown, where again we double-count the overlap.

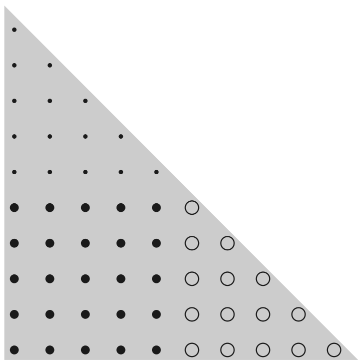


In the case $|r| \leq i$, then, we take

$$j = |r| + i; \quad k = i - |r|,$$

and again can verify the two requirements $T_j + T_k = r^2 + 2T_i$ and $k \geq 0$. In this case, $j - k = 2|r|$, which is strictly positive (and so $j > k$ as required) and even.

The ‘usually’ in the statement concerning overlaps of triangles is there because of the case $|r| = i$, which gives rise to a figure as shown (for $r = i = 5$). In this case there are no double-struck dots, but the mapping to (j, k) is still valid, and gives $k = 0$.



Inverting the mapping

We are given a pair (j, k) of non-negative integers, and wish to find corresponding integer r and non-negative integer i such that

$$n = T_j + T_k = r^2 + 2T_i.$$

If $j = k$, we take $r = 0$ and $i = j$ $(= k)$. Otherwise we proceed as follows.

If $j > k$, we let $(j_0, k_0) = (j, k)$, otherwise we must have $k > j$, and take $(j_0, k_0) = (k, j)$. Thus $j_0 > k_0$ always.

If $j_0 - k_0$ is odd, take

$$r_0 = \frac{j_0 + k_0 + 1}{2} > 0; \quad i = \frac{j_0 - k_0 - 1}{2} \geq 0,$$

where these expressions both result in an integer. Also, $r_0 > i$.

If $j_0 - k_0$ is even, take

$$r_0 = \frac{j_0 - k_0}{2} > 0; \quad i = \frac{j_0 + k_0}{2} > 0,$$

where, again, these expressions both result in an integer, but here $r_0 \leq i$.

Finally, if $j > k$ take $r = r_0$. Otherwise, we must have $k > j$, and we take $r = -r_0$ instead.

Example

For example, take $n = 276$. This integer has the following eight decompositions in terms of either $T_j + T_k$ or $r^2 + 2T_i$:

$$\begin{array}{ccccccc}
0 & + & 276 & = & 144 & + & 2 \times 66 \\
T_0 & & T_{23} & & (-12)^2 & & T_{11} \\[1ex]
45 & + & 231 & = & 36 & + & 2 \times 120 \\
T_9 & & T_{21} & & (-6)^2 & & T_{15} \\[1ex]
66 & + & 210 & = & 256 & + & 2 \times 10 \\
T_{11} & & T_{20} & & (-16)^2 & & T_4 \\[1ex]
105 & + & 171 & = & 4 & + & 2 \times 136 \\
T_{14} & & T_{18} & & (-2)^2 & & T_{16} \\[1ex]
171 & + & 105 & = & 4 & + & 2 \times 136 \\
T_{18} & & T_{14} & & 2^2 & & T_{16} \\[1ex]
210 & + & 66 & = & 256 & + & 2 \times 10 \\
T_{20} & & T_{11} & & 16^2 & & T_4 \\[1ex]
231 & + & 45 & = & 36 & + & 2 \times 120 \\
T_{21} & & T_9 & & 6^2 & & T_{15} \\[1ex]
276 & + & 0 & = & 144 & + & 2 \times 66 \\
T_{23} & & T_0 & & 12^2 & & T_{11}
\end{array}$$

References

[1] Bruce C. Berndt. *Number Theory in the Spirit of Ramanujan*, volume 34 of *Student Mathematical Library*. American Mathematical Society, 2006.